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A dependent bundles model for estimating stress concentrations in fibre–matrix composites

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This paper pursues the interest in characterizing stress concentrations in composites for the purpose of providing more reliable input into statistical theories about composite strength. A previous paper estimated stress concentration parameters in a fibre–matrix composite by numerical maximum likelihood, using the chain of bundles model of Harlow and Phoenix. An assumption of the model is that bundles are independent. Here, a model is proposed which allows for a dependence across bundles, usually due to the extent to which a fibre becomes unstressed in the region of a break. The model is demonstrated using experimental data on a carbon–epoxy hybrid composite, and arrays of tungsten-cored silicon carbide fibres embedded in resin. It is shown, via the method of maximum likelihood, how inferences may be made about stress concentrations and also parameters describing single fibre strength.

1. Introduction

This paper develops the ideas put forward previously [1]. The basic descriptions of models are the same, but the procedure here is different and a greater generalization is made. Wolstenholme and Smith [1] describe how the whole process of failure, rather than just the final strength, of a uniaxial composite may be modelled. Experimental data giving the positions and applied loads at which individual fibre failures occur may then yield numerical estimates of parameters describing the stress concentrations induced in those unfailed fibres which are close to failed fibres. Of particular interest is how the stress concentrations vary with inter-fibre spacing. The chain-of-bundles model [2–5] was used to describe the composite and shown to be effective in the estimation procedure, provided the approximation imposed by assuming sub-bundles to be independent is not too great a departure from the true situation. This depends largely on the length of fibre unstressed either side of a break being fairly small. This, in turn, creates overstressing in an equivalent length, the positively affected length (PAL), in neighbouring unbroken fibre segments. If the PAL is reasonably confined to one sub-bundle, then the dependency of sub-bundles is fairly small. However, it was shown in [1] that for some data, the choice of sub-bundle size can be very influential.

An alternative model is proposed here which effectively allows for dependence across sub-bundles when, in addition to the points of fibre failure being known, the length of consequent unstressed fibre around each break is also specified. The model was motivated by observations from a composite constructed from tungsten-cored silicon carbide fibres embedded in a flexibilized resin. When these fibres break, a release of energy sends a shock wave along the length of a fibre and results in sometimes quite considerable lengths either side of the break being multiply fractured and, therefore, unstressed (Fig. 1).

2. A "dependent bundles" model

2.1. Description

Suppose we have N parallel fibres each L "units" long. Consider each unit length of fibre to be an "element" whose strength is described by a Weibull distribution with shape and characteristic stress parameters as for unit length fibres.

The failure process may be described as a sequence of "events", where each event is an individual fibre break and the associated unstressing of a length of fibre either side of the break. Here, a fibre break will be called primary failure, and consequential unstressing, secondary failure.

The "status" of the $N \times L$ fibre elements at any point in the failure process can be described as stressed or unstressed and indicated by 1 or 0 entered in an $N \times L$ matrix. When an element is adjacent to unstressed elements in neighbouring fibres, it will be subject to an enhanced loading characterized, for example, by a stress concentration factor $k = 1 + u(r)$, where $u(r)$ is a function of r, the number of unstressed elements which are "adjacent", in the sub-bundle sense. When the applied stress is x, the fibre experiences stress kx.

The joint probability, or likelihood function, for the given set of events may be obtained. When the ith break occurs at stress x_i, a contribution to the likelihood function must be calculated from all elements which were unfailed immediately prior to the ith break

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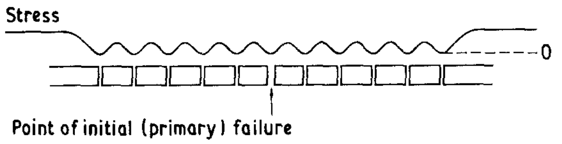


Figure 1 Schematic representation of multiply fractured fibre and resulting stress distribution.

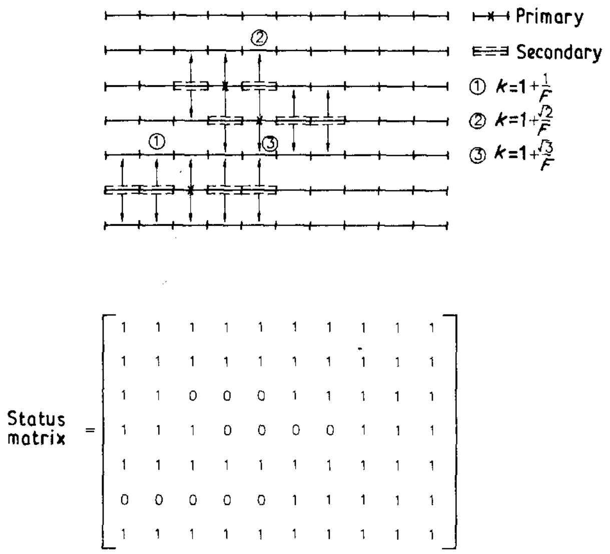


Figure 2 An example of the "dependent bundles" model and associated status matrix.

and also the density associated with the failing element. In the light of the recorded primary and secondary failure of certain elements, the status matrix is then adjusted, and the stress-concentration factors of surviving elements adjusted. An example of the "dependent bundles" model and associated status matrix is shown in Fig. 2, where $u(r)$ has been taken as $r^{\frac{1}{2}}/F$, F being a load-sharing parameter, following Bader and Pitkethly [6].

2.2. Mathematical details

Let $\boldsymbol{T}^{(n)} = \{t_{i,j}^{(n)}\}$ = status matrix, and $\boldsymbol{K}^{(n)} = \{k_{i,j}^{(n)}\}$ = stress concentration factor matrix at the nth break.

Let $f(x)$, $F(x)$ = Weibull density and distribution functions for the strength of unit length fibres, $\boldsymbol{G}^{(n)} = \{g_{i,j}^{(n)}\}$ = matrix of element likelihoods after n breaks, and m = total number of breaks.

Initially $t_{i,j} = k_{i,j} = 1$ for all i, j. Let the stress at which the first break occurs be x_1. Then $g_{i,j}^{(1)}$ for all elements, except that where the break occurred, is the probability that a unit element survives stress x_1. This probability is $1 - F(x_1)$, or equivalently $S(x_1)$, where $S(.)$ is the survivor function. When an element has failed at a known stress x, the probability is given by $h(x)S(x)\mathrm{d}x = f(x)\mathrm{d}x$, where $h(.)$ is the hazard function. The differential $\mathrm{d}x$ is not a function of unknown parameters and will be omitted for the moment. So for the element containing the first break at stress x_1, $g_{i,j}^{(1)} = f(x_1)$.

The contribution to the likelihood function is then $\prod_{i,j} g_{i,j}^{(1)}$ but it is more convenient to deal with the log-likelihood function, $\sum_{i,j} \log(g_{i,j}^{(1)})$.

In general, at the nth break, at stress x_n, in the bth element of the rth fibre

$$g_{i,j}^{(n)} = t_{i,j}^{(n)} S(k_{i,j}^{(n)} x_n)/S(k_{i,j}^{(n-1)} x_{n-1}) \qquad (1)$$

except

$$g_{b,r}^{(n)} = s_{b,r}^{(n)} f(s_{b,r}^{(n)} x_n)/S(s_{b,r}^{(n-1)} x_{n-1}) \qquad (2)$$

and the contribution to the log-likelihood function is $\sum_{i,j} t_{i,j}^{(n)} \log(g_{i,j}^{(n)})$.

In practice, a fibre element's total contribution can be calculated when the element has primary failure, secondary failure or survived the maximum applied stress.

Define $\boldsymbol{G}$ to be the matrix of element total likelihoods, then for

(a) primary failed elements

$$g_{i,j} = k_{i,j}^{(n)} f(k_{i,j}^{(n)} x_n) \qquad (3)$$

(b) secondary failed elements

$$g_{i,j} = S(k_{i,j}^{(n)} x_n) \qquad (4)$$

(c) surviving elements

$$g_{i,j} = S(k_{i,j}^{(m)} x_{\max}) \qquad (5)$$

where $x_{\max}$ is the maximum applied stress. The overall log-likelihood

$$\log L(\boldsymbol{\theta}; \boldsymbol{x}) = \sum_{i,j} \log(g_{i,j}) \qquad (6)$$

and may be maximized numerically with respect to unknown parameters $\boldsymbol{\theta}$.

2.3. Assumptions

The analysis implicitly assumes that the strength distribution of unit length fibres may be defined, for whatever we may choose a "unit" to mean. This is also true for the chain-of-bundles model, and inherent in both is that the principle of weak-link scaling applies, and hence the reliance to a large extent on the Weibull distribution. Strength testing fibres of length 1 mm or less is extremely difficult, so invariably predictions are made from tests at longer lengths. Alternatively, we can, given enough computing time, estimate the unit length parameters by including them in the vector of unknown parameters $\boldsymbol{\theta}$.

The assumption that load redistribution is uniform, on a sub-bundle basis, is a slight approximation, but the model is versatile enough to allow stress concentration factors to be not only a function of r, but also, for example, distance from a primary failure point. So far, a "local" load-sharing rule has been assumed, but there is no reason why other rules should not be used, if thought appropriate.

2.4. More than one failure at a given stress

The model as described has not taken account of the possibility that more than one failure may be recorded at any given stress. Such simultaneous failures may be regarded as independent provided their regions of secondary failure do not overlap. Dependency will, in fact, only be possible when the region of overlap contains at least one of the primary failures. A procedure has been devised which automatically sorts out the dependence by monitoring changes in load-scaling factors, and will be discussed later. First, though, it needs to be clear how to treat elements involved in simultaneous or group failure. This will involve sums and products of probabilities, and a $g_{i,j}$ cannot be simply defined for all i,j involved. The procedure adopted is to calculate the joint probability for all elements involved in the group failure and assign this probability to just one of the $g_{i,j}$ and set all the other $g_{i,j} = 1.0$.

2.5. Dependent failure probabilities

Consider two neighbouring fibres with breaks occurring at the same stress, x (Fig. 3). It will be assumed that the combined failure is initiated by one or other of the two breaks. Let element j in fibre i have stress concentration factor, $k_{i,j}$, and denote by $k'_{i,j}$ the new factor as a result of the other fibre failing first. Let the order of failure be fibre 1 then fibre 2. Then contributions to the joint probability for elements 4, 5 and 6 in fibre 1 are

$$a_{1,4} = S(k_{1,4}x) \qquad (7)$$

$$a_{1,5} = k_{1,5}f(k_{1,5}x) \qquad (8)$$

and

$$a_{1,6} = S(k_{1,6}x) \qquad (9)$$

In fibre 2 elements 7 and 8 are both unaffected by the failure of fibre 1, so $a_{2,7} = S(k_{2,7}x)$ and $a_{2,8} = S(k_{2,8}x)$.

Calculation of the probability contributions for elements 5 and 6 is based on knowing that the break in element 6 occurs somewhere between stress $k_{2,6}x$ and $k'_{2,6}x$. The probability of failing between these two stresses is

$$S(k_{2,6}x) - S(k'_{2,6}x) = a_{2,6} \qquad (10)$$

The exact failure stress is not known which poses the question of how to define a suitable survival probability for the secondary failed element 5 (Fig. 4).

If element 6 fails at stress y, and element 5 is under stress z at the time, then it is desirable to be able to define z in terms of y. The term $a_{2,5}$ is then given by $S(z)$. A similar set of probabilities, $b_{i,j}$ say, is required for the order of failure 2 then 1, so the total likelihood contribution for the seven elements involved is

$$\prod_{j=4}^{6} a_{1,j} \prod_{j=5}^{8} a_{2,j} + \prod_{j=4}^{6} b_{1,j} \prod_{j=5}^{8} b_{2,j} = \prod_{j=4}^{6} g_{1,j} \prod_{j=5}^{8} g_{2,j} \qquad (11)$$

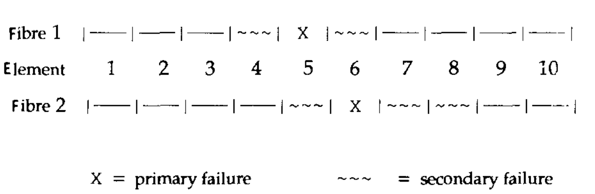


Figure 3 Neighbouring fibres with breaks occurring at the same stress.

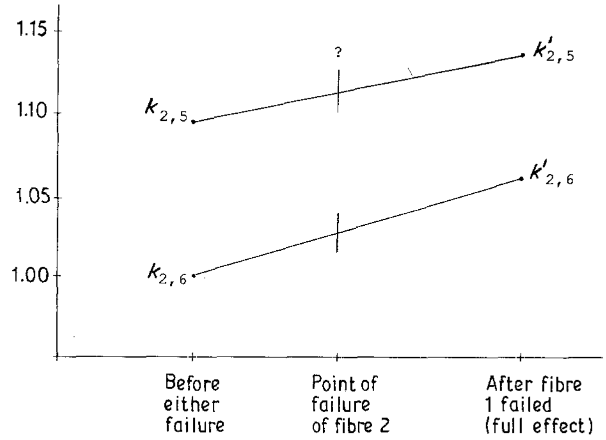


Figure 4 Changes in load concentration factors in the course of group failure.

As stated in the previous section, for convenience this joint probability is assigned in total to just one of the $g_{i,j}$, involved, other $g_{i,j}$ set to 1.0.

2.6. The treatment of secondary failure

Consider the general example shown in Fig. 5. Consider the case when fibre 1 fails first, followed by fibre 2. Let $k_{i,j}$ and $k'_{i,j}$ be defined as in 2.5. The likelihood contributions are:

for fibre 1,

$$S(k_{1,j}x)k_{1,j}f(k_{1,j}x)\prod_{i \neq j}^{n_1} S(k_{1,i}x) \qquad (12)$$

where the jth element is the primary failure,

and for fibre 2,

$$\int_{k_{2,j}x}^{k'_{2,j}x} S(z_i)f(y)\,\mathrm{d}y \qquad (13)$$

where $z_i = g_i(y)$.

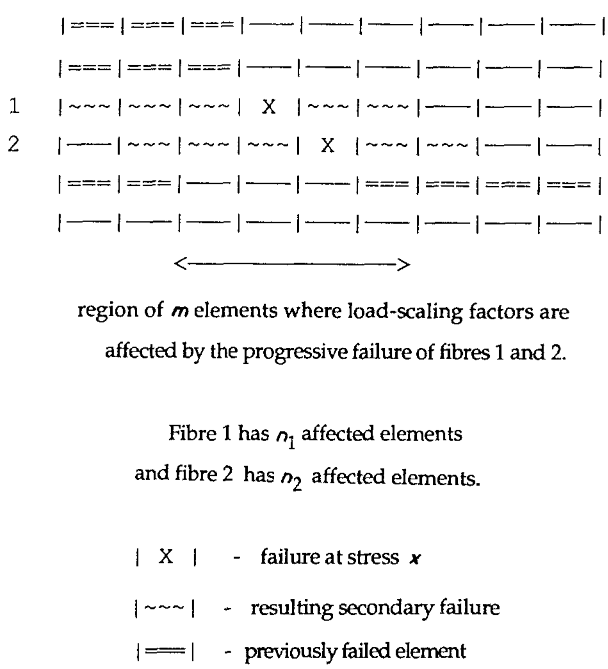


Figure 5 A general example of group failure.

Various definitions for $g_i(y)$ are possible. The "conservative" and simplest approach is to assume that nothing is really known about z_i except that it must be at least $k_{2,i}x$ – see Fig. 4. Then Expression 13 becomes

$$\left[\prod_{i \neq j}^{n_2} S(k_{2,i}x)\right][S(k_{2,j}x) - S(k'_{2,j}x)] \qquad (14)$$

This means that the interactions in group failure are reflected solely in the primary failures.

An alternative, more precise, interpretation of $g_i(y)$ is described in the Appendix. For the data considered here the difference made to the overall results is small. The above simple approach will be adopted in the main body of this paper.

3. Applications

3.1. Glass–carbon hybrids

To test the performance of the dependent bundles model, it was applied to some of the data examined previously [1]. The material consisted of 1000-ply carbon fibre tows embedded as linear arrays in a glass–epoxy matrix. The arrays of 7 tows were placed at various spacings from 1.5 mm tow centre to tow centre, down to tows just touching. A tow was treated as a single fibre and using the chain-of-bundles model, inferences were made about the load-sharing parameter, F, at different tow spacings. Some difficulty was encountered in the case of tows at 1.0 mm spacing, where it was found that varying the sub-bundle length produced irregular behaviour in the resulting estimates of F. The sub-bundle length is designed to correspond approximately to the PAL, and as this length is increased it is expected that the estimate of F will increase, i.e. the stress concentrations are lower, because they are effective over a longer region. There were five samples of length 100 mm, and as they were all cut from the same sheet of material, the data were combined in order to estimate what were assumed to be the common characteristics across all samples.

The Weibull strength parameters were taken to be shape parameter, w, = 30 and unit length characteristic stress, α, = 4.12 GPa, though with a considerable degree of uncertainty. This yielded for various choices of sub-bundle length, d, the results shown in Table I. The unsatisfactory nature of these results was put down partly to some unexplained strength variation in the material, and partly due to the surprising degree to which the choice of d altered the apparent structure of the arrays.

In the dependent bundles model, the choice of unit element length, d, is not of any particular physical significance, but should be designed to combine numerical accuracy with acceptable computation time. It is, as already stated, most important that the strength function $f(x)$ for fibres of length d is reliably defined.

TABLE I

d	$\hat{F}$	95% confidence interval
1.0	80	[57, 122]
2.0	48	[42, 56]
2.5	115	[85, 175]
4.0	67	[58, 78]

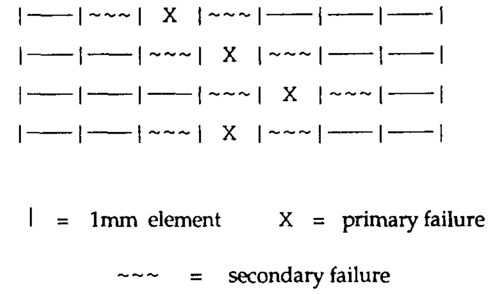


Figure 6 Primary failures with one ineffective element on either side.

The position of fibre breaks in these data were given to the nearest millimetre, so taking this as the value for d, a direct comparison could be made with the chain-of-bundles results for $d = 1.0$. The data has to be presented in strain or stress order (assuming a constant fibre modulus) with dependent simultaneous failures indicated. Initially no regions of secondary failure were indicated so the bundle only consisted of elements which had a primary failure or which had survived to the maximum stress. Maximizing the dependent bundles model likelihood with respect to F gave exactly the same results as those shown in Table I for $d = 1.0$. The maximized log-likelihood was -913.75.

Perhaps to cast more light on these data, and to demonstrate the wider application of this model, the data were amended to allow for an "ineffective" length either side of each fibre break, which would be equivalent to a region of secondary failure. It is only possible to guess at a suitable value for the ineffective length, – 5 or so fibre diameters has been suggested by the works of Rosen, and Zweben [7–9]. Here, with the "fibres" being tows of approximately 0.3 mm diameter an ineffective length of 1.5 mm or more would be suggested. There were instances where breaks as close as 1 mm, were recorded but with recording accuracy to the nearest millimetre this could be a distance of up to 2 mm. Overall it seems reasonable to adopt the strategy of assigning one "element" either side of the breaking element to be consequently ineffective, as illustrated in Fig. 6. This now allows for possible dependence across sub-bundles as well as preventing the erroneous linking of independent failures which just happen to fall in the same sub-bundle.

The incidence of simultaneous failure here is small. In the total 371 breaks, choice of approach to simultaneous failure will make little difference. Once again taking $w = 30$, estimates of F were calculated for various α in order to achieve the best estimates for F and α combined (Table II). The results marked ** are where the negative log-likelihood is at a minimum, or equivalently the log-likelihood at a maximum, and the value, when compared with the earlier value of 913.75, indicates that this is a more plausible scenario than restricting the effect of fibre breaks to just one element. However, the information about elements rendered

TABLE II

α	$\hat{F}$	$-\log L$	Conf. int.
4.12	122	886.55	[90, 179]
**4.11	136	885.09	[100, 205]
4.10	153	885.16	[110, 233]
4.09	172	886.90	[122, 266]

ineffective is more vague which may reduce the precision of estimation.

It had been observed that there was a clear difference in the strength of these samples compared to other experimental data, so it may be informative to try to estimate all three parameters, F, α, and w, though the calculation involved is lengthy and it is not practical to obtain estimates to a great degree of accuracy. However, Table III gives a good idea of the region in which the joint estimates lie. Confidence regions may be calculated using the large-sample properties of maximum likelihood estimates. If $\hat{\boldsymbol{\theta}}$ are the maximum likelihood estimates of unknown parameters $\boldsymbol{\theta}$ then an α% confidence region is approximately given by all $\boldsymbol{\theta}^*$ such that

$$2\log[L(\hat{\boldsymbol{\theta}};x)/L(\boldsymbol{\theta}^*;x)] < \chi^2_{(p,\alpha)} \qquad (15)$$

where p is the dimension of θ and $\chi^2_{(p,\alpha)}$ is the α% point of the Chi-squared distribution with p degrees of freedom. So in Table III the confidence region shown is where the log-likelihood value is within $0.5\chi^2_{(3,\,0.95)}$ of the optimum value.

Approximately: $\hat{F} = 115$, $\hat{w} = 24$, $\hat{\alpha} = 4.18$, with $-\log L = 872.45$, the latter very significantly lower than with $w = 30$. A smaller w and hence greater variability might be expected where some experimental deviation from the norm is present.

What is clear is that regardless of the true strength parameters, F is of the order 90–140, which reflects a small degree of load-sharing, rather less than in the case of fibres at 0.5 mm spacing, where the estimated F was 15–20. It is also clear that the model works and has overcome certain difficulties encountered with the chain-of-bundles model.

TABLE III Maximum likelihood estimates of F over a grid of (α, w) values, with corresponding negative log-likelihood values. The best estimate of F for a given w is indicated by *, and the best combination of (α, w, F) is shown by **

α	w					
	30	28	26	24	22	20
4.09	172					
	886.9					
4.10	153	179				
	885.2	883.9				
4.11	136*	160				
	885.1	880.5				
4.12	122	144				
	886.5	878.8				
4.13		129*	154			
		878.5	877.3			
4.14		116	139			
		879.6	875.1			
4.15			126*	152		
			874.2	877.9		
4.16			114	138		
			874.5	875.0		
4.17				126		
				873.2		
4.18				115**		
				872.4		
4.19				105		
				872.7		
4.20					119	
					875.0	
4.21					110	
					873.8	
4.22					101*	
					873.6	
4.23					93	
					874.1	
4.24						
4.25						101
						878.6
4.26						94*
						878.1
4.27						87
						878.3

*The figures in the broken line frame show the confidence region (see above).

3.2. Tungsten-cored silicon carbide fibres in resin

3.2.1. Experimental procedure

The fibres were 0.1 mm diameter silicon carbide which had been manufactured by chemical vapour deposition of SiC on to a heated 0.012 mm diameter tungsten wire core. The composite consisted of a linear array of 10 parallel fibres set in a flexibilized epoxy resin matrix. The distance between fibres (centre-to-centre) was varied between 2 and 10 fibre diameters i.e. 0.2–1.0 mm. The length of the fibres was 200 mm. The experimental work has been previously reported by Clarke and Bader [10]. The aim of the experiment was to model composite failure phenomena, using single fibres of relatively large diameter to make handling and construction easier.

An unusual feature of this composite was the extended ineffective region surrounding each fibre break. It is thought that shockwaves induced by the release of energy at failure result in the observed multiple secondary fracturing and debonding from the matrix. The length of the affected region was variable, and was probably a function of stress level and length of surrounding unbroken fibre. Observation through crossed polarizers allowed photographs to be taken as the sample coupons were loaded, and from these photographs the positions of breaks could be recorded as well as the lengths of associated regions of secondary failure. Where a photograph contained two or more breaks, these had to be regarded as simultaneous failures.

All coupons for a given fibre spacing came from the same sheet of material, which was large enough to produce 14 parallel coupons but only a sample of these were tested. The sample number indicates the position of the coupon on the sheet. The spacings 2, 4, 6, 8, 10 fibre diameters are indicated by the letters N, O, P, Q, R. A feature of certain of the test coupons was that at higher stresses, cracks in the resin started to appear which almost certainly creates a new load redistribution situation, but is of unknown nature. In view of this it was decided initially to truncate those data sets at the point when the first resin crack appeared. Figs 7–11 show schematic diagrams of the failure patterns and Table IV gives details of the minimum and maximum breaking strains and the appearance of cracks.

3.2.2. *The strength distribution of single fibres*

Since each group of coupons was cut from the same sheet, it was to be expected that all *N* data would have similar characteristics, all *O* data would be similar, and so on, but that these characteristics might differ slightly due to experimental variations in laying-up, curing, etc. Any major component of variation would be attributable to the different fibre spacings.

It is very clear from Table IV that as the fibres are placed closer together, i.e. as less resin separates them, there is a lowering of the inherent strength of the fibre. Early failures point to this conclusion as they cannot be affected by any load-sharing at this stage. Sets *N*, *O* and *P* would appear to be similar, and then a clear distinction between these sets and *R* and *Q*.

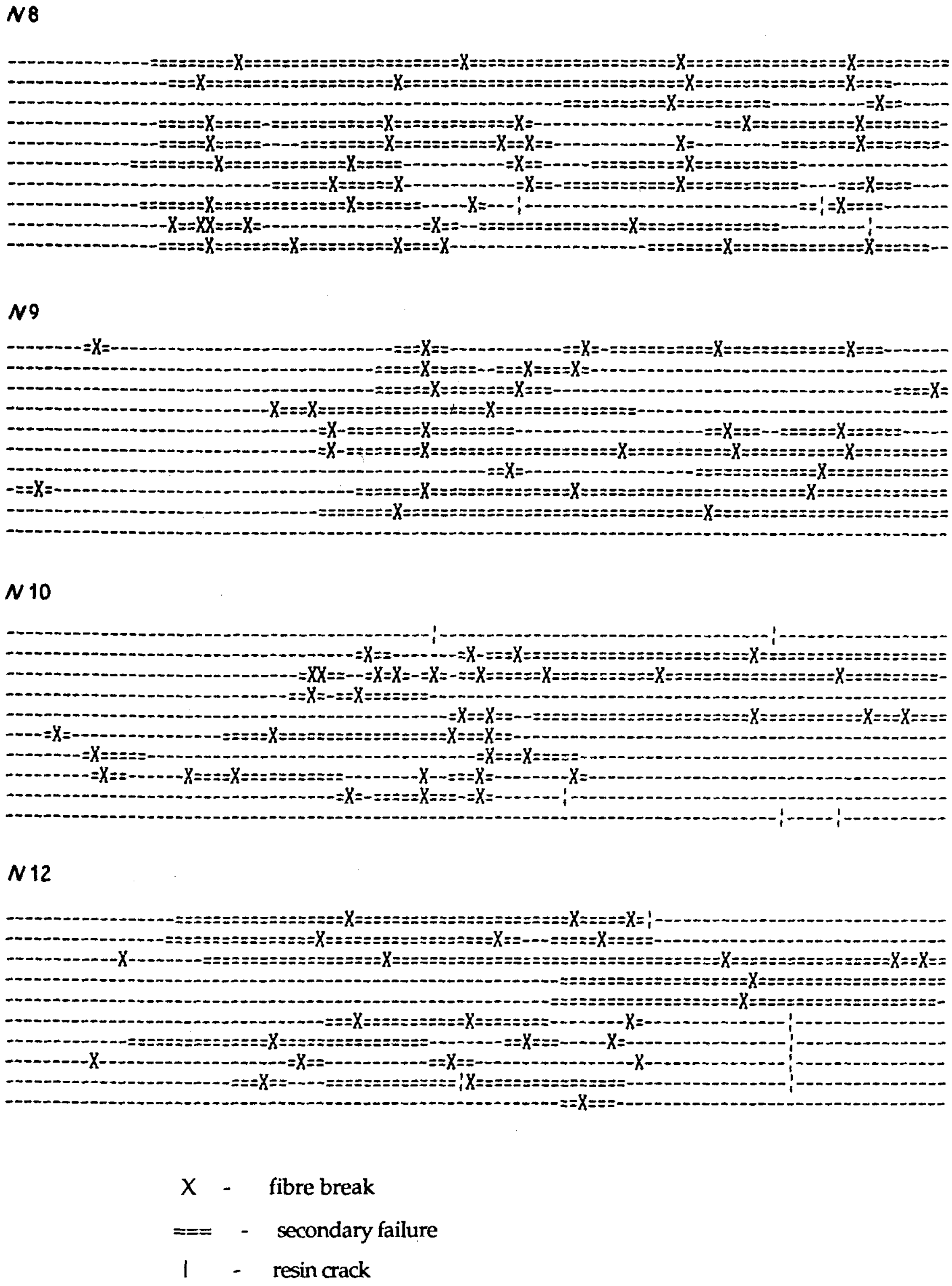


Figure 7 Patterns of breaks in silicon carbide arrays in resin with fibres at 2 diameter (0.2 mm) centre-to-centre spacing.

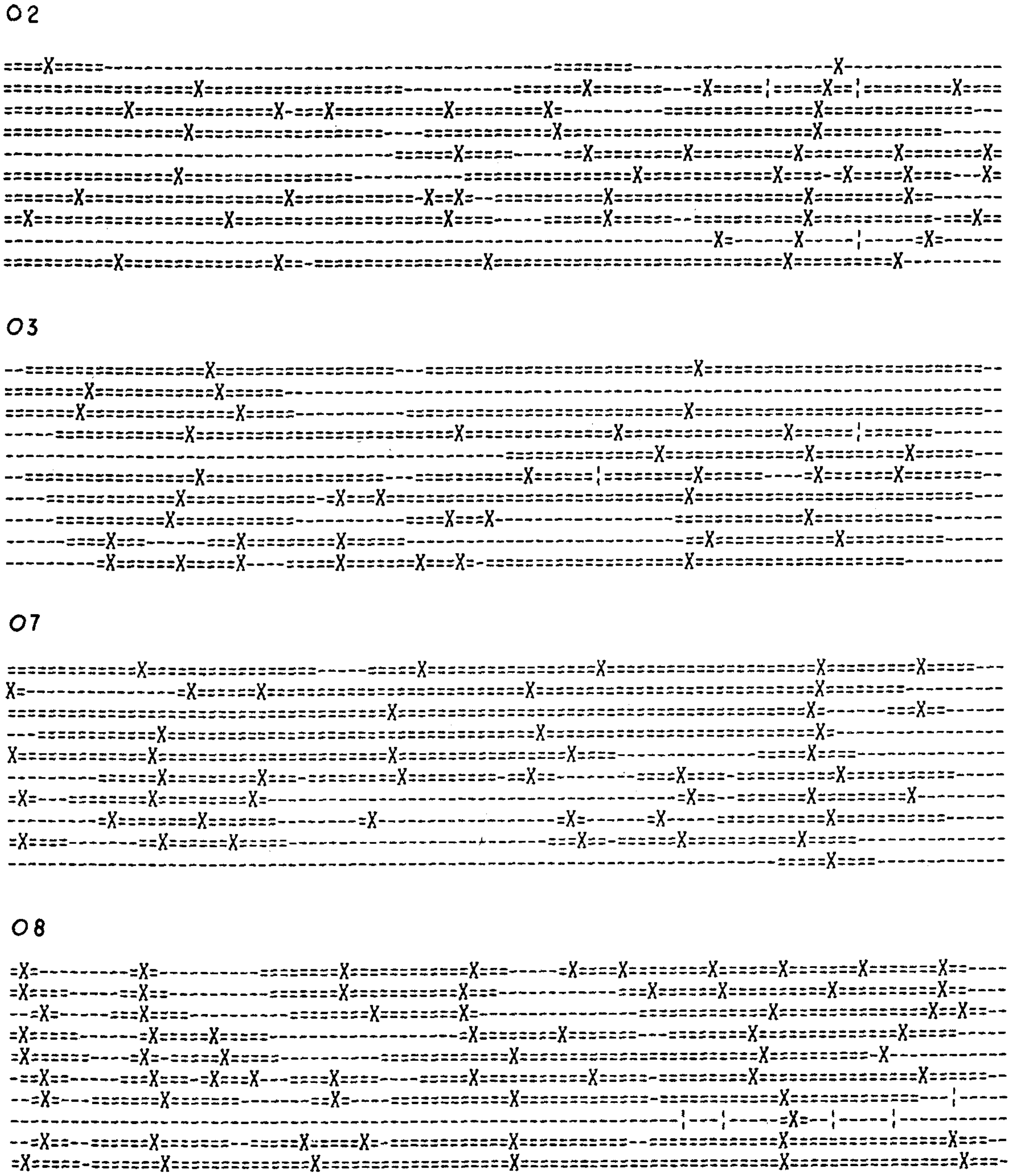


Figure 8 Patterns of breaks in silicon carbide arrays in resin with fibres at 4 diameter spacing.

Clarke [11] tested single 200 mm fibres in resin and estimated the parameter w for the Weibull strength distribution to be 13.9, and characteristic strain for 200 mm to be 0.98%. The modulus for this fibre is 347 GPa and by weak-link scaling this implies the value 4.98 GPa for the parameter α, the characteristic stress for length 1 mm. Similar tests on 100 mm and 50 mm gauge lengths indicated that a Weibull model of this order fitted quite well, though the number of observations in each case was small (13 or 14).

Under weakest-link theory the stress of the first failure in 10 fibres length 200 has a Weibull distribution, parameters w and $\alpha(200 \times 10)^{(-1/w)}$. Simple calculation shows the R and Q data to be consistent with the parameters supplied, but the N, O and P data to be clearly inconsistent. It was therefore necessary to investigate the true strength distribution before applying the dependent bundles model to investigate stress concentrations.

3.2.3. Estimation of unit length strength distribution

The early stages of failure in an array can be considered to be largely independent of the load-sharing aspect. Where the first failure recorded in an individual fibre is one where neighbouring fibre elements have not yet failed, then this can be regarded as an independent observation of failure of a fibre of the given length. If failure might have resulted from load passed from neighbouring fibres then a right-censored observation may be recorded.

Given n fibres of length L, with m initial failures at stresses $x_1, x_2, \ldots, x_m$ and $n - m$ right-censored observations at stresses $y_1, y_2, \ldots, y_{n-m}$, then the likelihood function is of the form

$$\prod_{i=1}^{m} f_L(x_i) \prod_{j=1}^{n-m} S_L(y_j) \qquad (16)$$

where $f_L(.)$ and $S_L(.)$ are the probability density and

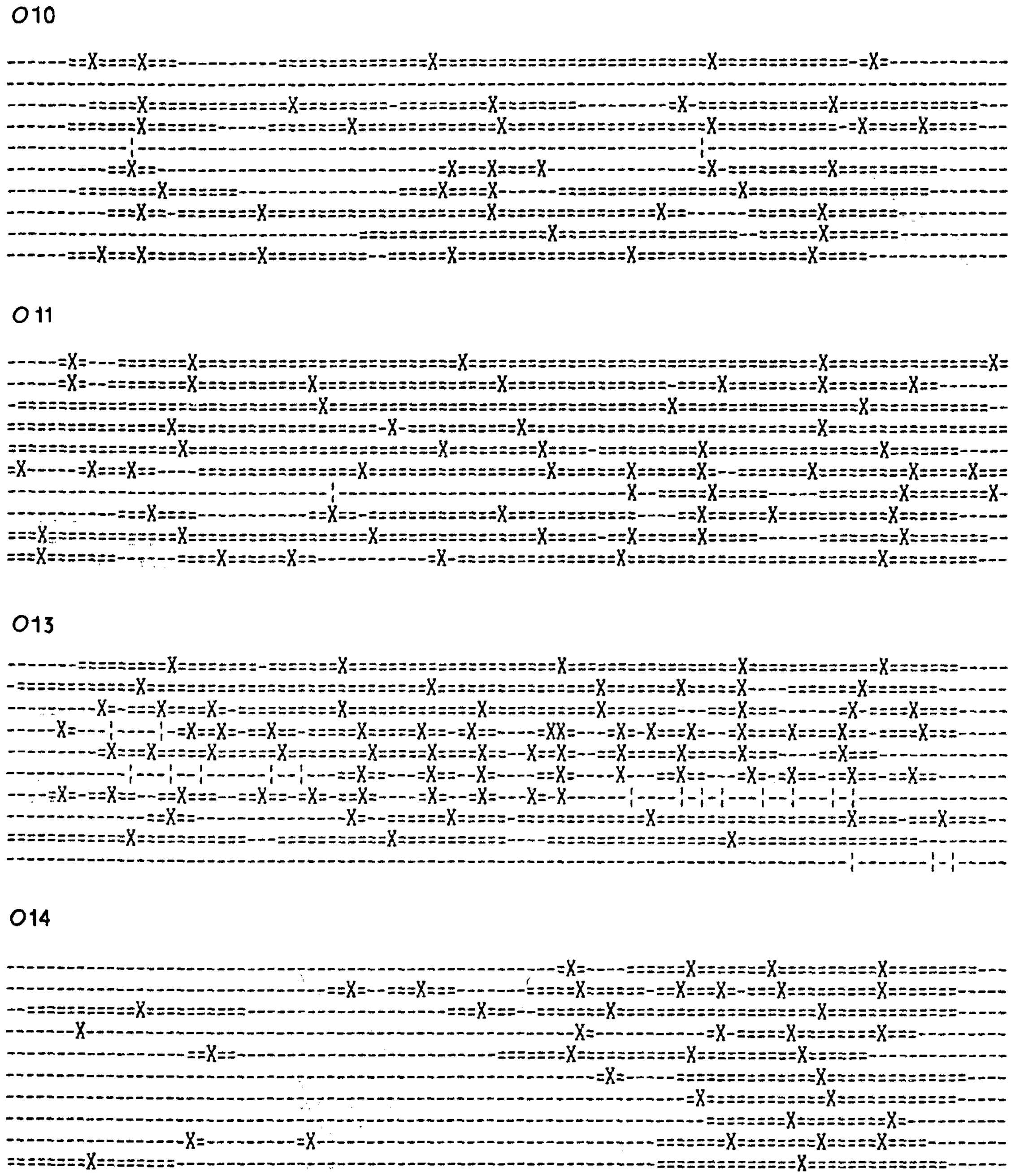


Figure 9 Patterns of breaks in silicon carbide arrays in resin with fibres at 4 diameter spacing.

survivor functions for the strength of fibres length L. The asymptotic properties of the maximum likelihood estimators in the presence of general independent censoring mechanisms are not known exactly, but Kalbfleisch and Prentice [12] state that certain arguments imply that the usual results hold under fairly mild conditions on the hazard and any covariates.

Taking $f_L(x_i) = w/x_i L(x_i/\alpha)^w \exp\{-L(x_i/\alpha)^w\}$; $S_L(y_j) = \exp\{-L(y_j/\alpha)^w\}$, then the maximum likelihood estimate of α is given by

$$\hat{\alpha} = \left[L\left(\sum_{i=1}^{m} x_i^w + \sum_{j=1}^{n-m} y_j^w\right)\Big/ m\right]^{1/w} \quad (17)$$

For given w, the presence of right-censored observations will tend to increase the estimate of α, and reduce the precision of estimation. The maximum likelihood estimate of w has to be obtained iteratively. Nelson [13] gives a survey of numerical methods appropriate to such problems.

The result of this point estimation for the initial failures in the silicon carbide data is shown in Table V, with standard errors shown in brackets. Confidence regions based on the χ^2 approximation are shown in Fig. 12. There is a clear distinction between fibres at 2, 4 and 6 diameter spacings and the more widely spaced groups, which appear to conform reasonably to the earlier single fibre experiments. The question now arises as to whether a Weibull distribution fits these data well enough for a corresponding Weibull distribution to be assumed in each case for unit length strength distribution.

The usual Weibull probability plot requires $\log(x_i)$ to be plotted against $\log[-\log P(x_i)]$, where $P(x_i)$ is the estimated survivor function. Let $\{x_i\}$, the observed

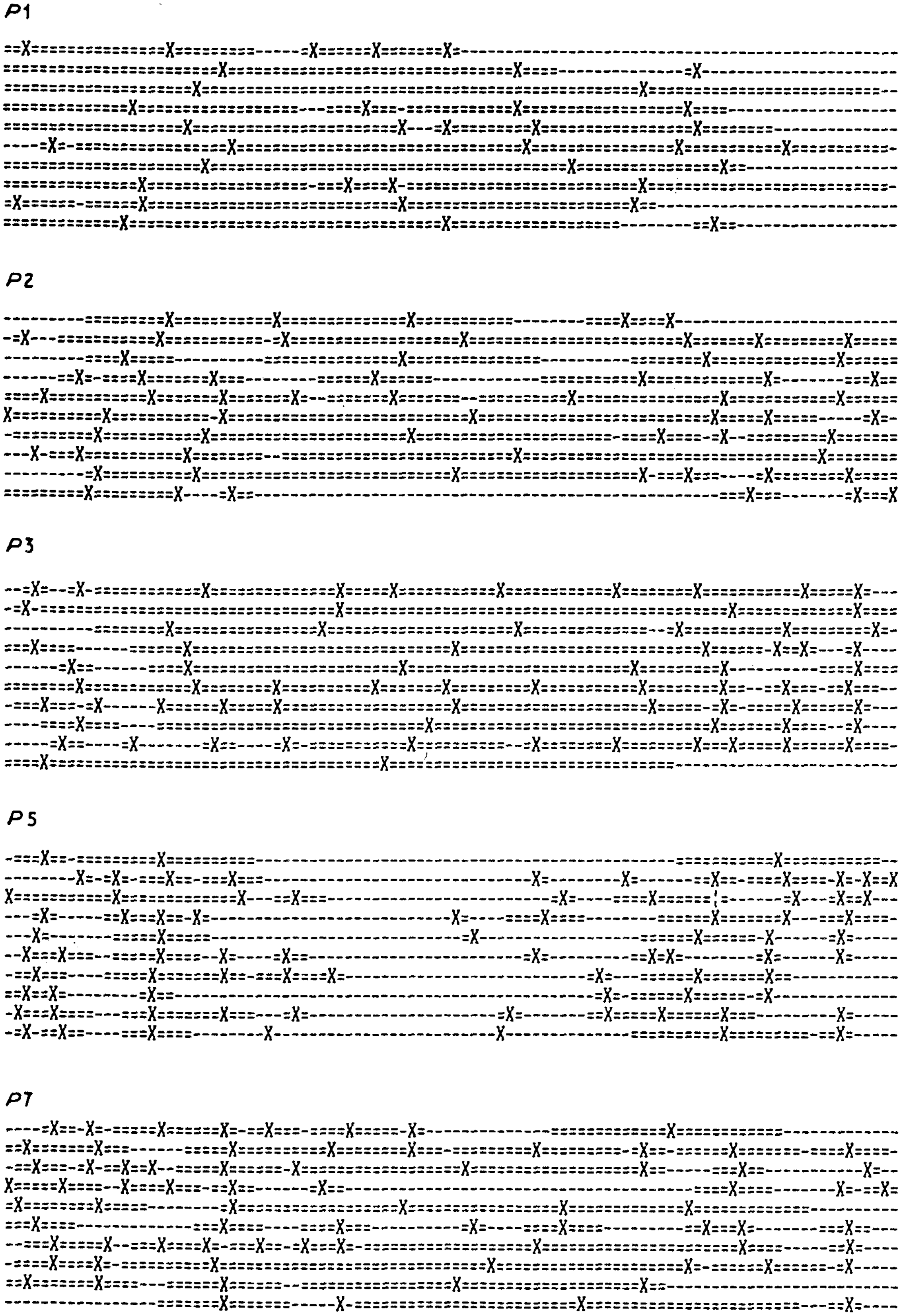


Figure 10 Patterns of breaks in silicon carbide arrays in resin with fibres at 6 diameter spacing.

failure stresses be such that $x_1 < x_2 < x_3, \ldots, < x_n$. $P(x_i)$ is the observed proportion of observations greater than $x_i = 1 - q_i$. The value for q_i may be, for example, $(i-1)/n$, $i/(n+1)$, $(i-0.3)/(n+0.4)$, $(i-3/8)/(n+1/4)$, $(i-0.44)/(n+0.12)$, $(i-1/2)/n$. Cunnane [14] reviews such estimators and concludes that there is little effective difference between the latter three, due respectively to Blom [15], Gringorten [16], and Hazen [17]. The Hazen formula is, in general, the least biased for larger samples, and this has been adopted in this work for uncensored samples.

In the case of progressively right-censored data, the function must be modified to take account of the censoring. The appropriate product limit estimator was introduced by Kaplan and Meier [18], and is described by Kalbfleisch and Prentice [12] for

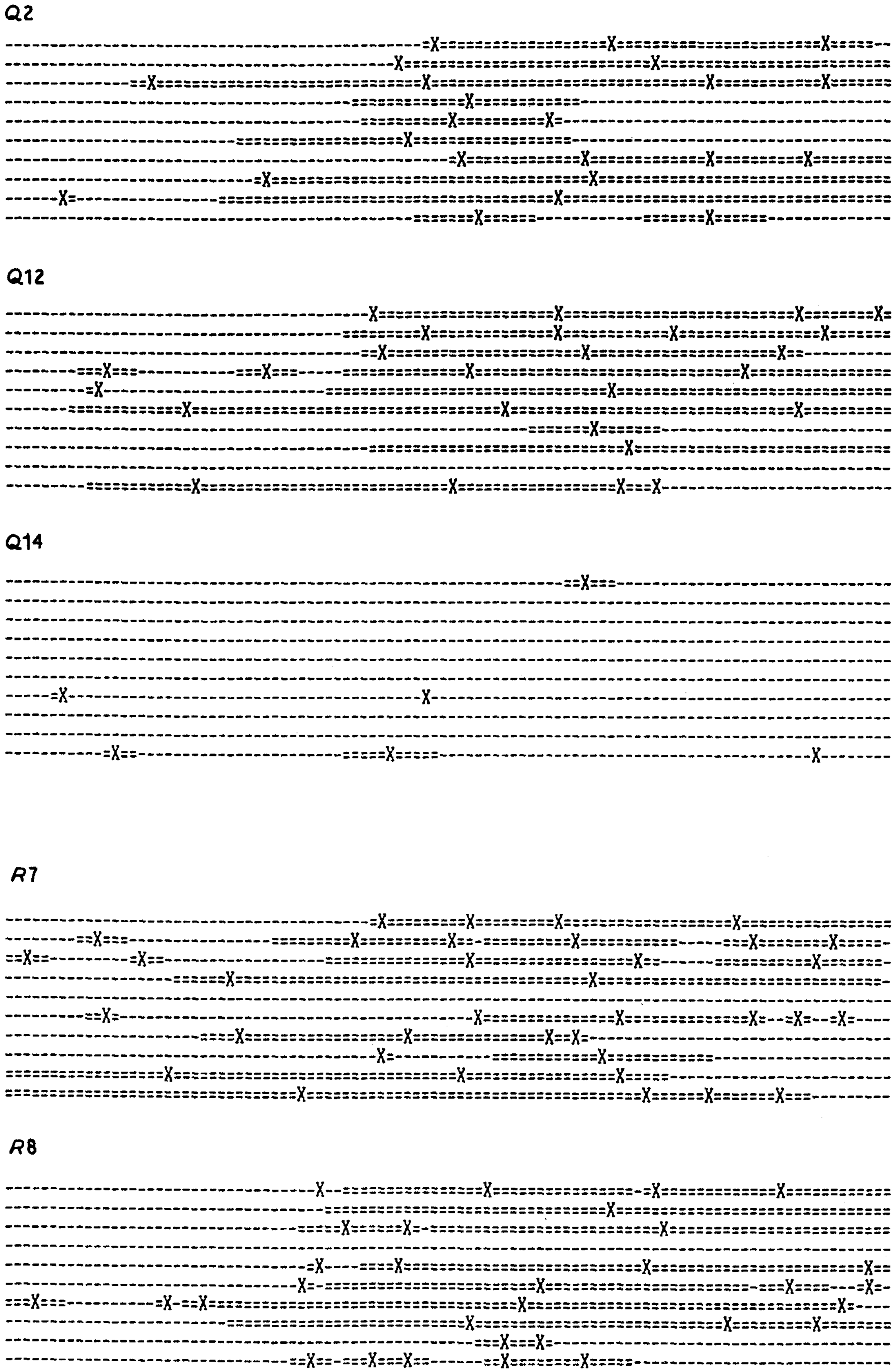


Figure 11 Patterns of breaks in silicon carbide arrays in resin with fibres at 8 diameter (Q) spacing and 10 diameter (R) spacing.

example. With the estimated survivor function so defined, Weibull probability plots were produced for each set of initial failures (Fig. 13). The distinction between close and wider spacings is still evident, and there is no real evidence of a significant departure from the Weibull model. The Q and R data conform well, N, O, and P a little less so. Physical explanations for this difference will be discussed later. It will be assumed for the moment that a Weibull model fits the unit length strength in each case, and that the most appropriate parameters to use are those given by the initial failures.

TABLE IV Maximum and minimum strain (%)

Data set	Strain			No. of breaks		No. of resin cracks
	Min.	1st resin crack	Max.	Before cracks	Total	
$N8$	0.694	0.870	0.919	38	46	3
$N9$	0.681		0.809		31	
$N10$	0.485	0.912	1.012	21	36	5
$N12$	0.555	0.852	0.966	16	26	6
$O2$	0.510	0.899	0.999	42	49	4
$O3$	0.577	0.953	1.054	32	39	2
$O7$	0.625		1.173		46	
$O8$	0.603	1.026	1.140	58	66	5
$O10$	0.673	0.983	1.132	24	39	2
$O11$	0.535	1.091	1.098	53	57	1
$O13$	0.521	0.699	1.003	6	79	18
$O14$	0.640		1.031		37	
$P1$	0.755		1.123		38	
$P2$	0.645		1.126		62	
$P3$	0.628		1.019		71	
$P5$	0.608	0.616	1.045	3	80	1
$P7$	0.643		1.057		76	
$Q2$	0.925		1.050		23	
$Q12$	0.870		1.083		26	
$Q14$	0.820		0.955		6	
$R7$	0.898		1.110		36	
$R8$	0.956		1.087		31	

TABLE V

Data set	No. of uncensored obs.	No. of censored obs.	$\hat{w}$	$\hat{\alpha}$
N	28	12	8.6 (0.30)	5.376 (0.12)
O	69	11	6.1 (0.14)	6.848 (0.14)
P	45	5	7.9 (0.23)	5.572 (0.11)
Q[a]	28	12	14.8 (0.51)	4.885 (0.06)
R	17	3	16.7 (0.78)	4.875 (0.07)

[a] The Q data are from four coupons. One was omitted from later analysis because it was thought suspect due to possible bending, but the initial failures were felt to be valid observations.

3.2.4. Inference about F

The results obtained on maximizing the dependent bundles likelihood function with respect to F, using the supplied w and α for each fibre spacing are shown in Table VI. The results marked ** for the Q and R data are those obtained when the earlier strength parameter estimates from single fibre tests were used, i.e. $w = 13.9$, $\alpha = 4.98$. It makes little difference which are used, but it happens that the likelihood is always a little higher with the single fibre estimates, so these are used in all subsequent analyses.

The results are fairly satisfactory. The estimates of F increase as the fibre spacing increases, and the load-sharing relatively small at the 6-diameter spacing. It is generally expected that beyond 5 fibre diameters the stress transfer should be very small, and certainly at 8 or 10 diameters we would expect the confidence interval for F to stretch to infinity.

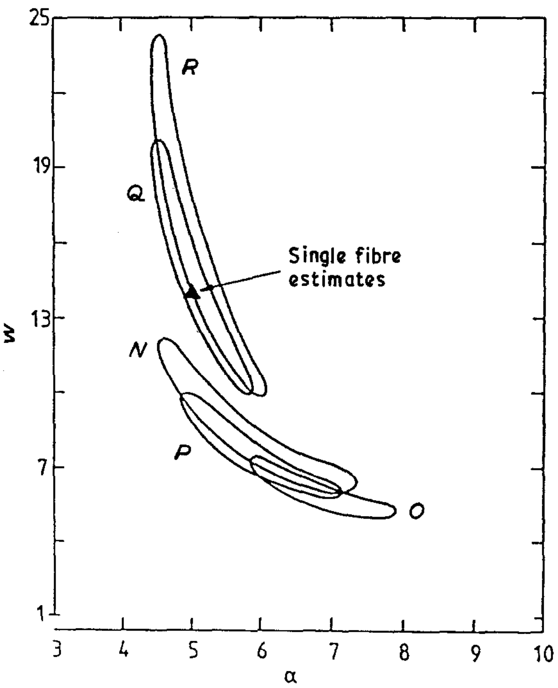


Figure 12 95% confidence regions for the Weibull parameters, based on initial failures.

Various modifications to the model were tried aimed at making the load redistribution as realistic as possible. A particular feature of these samples is the high percentage of the total fibre length rendered inoperative by secondary failure. This produces a not uncommon situation where surviving fibres have a high number of failed neighbours and hence a high load concentration on a $1 + r^{\frac{1}{2}}/F$ basis. There is little

TABLE VI

Data set	Fibre spacing (diameters)	w	α	$\hat{F}$	Min. − log$L$	Conf. int
N	2	8.6	5.38	12	549.51	[10, 17]
O	4	6.1	6.85	30	1519.53	[23, 46]
P	6	7.9	5.57	176	1151.02	[101, 371]
Q	8	14.8	4.89	290	280.78	[81, 4650]
**Q		13.9	4.98	270	280.70	[76, 4300]
R	10	16.7	4.88	170	312.76	[68, 890]
**R		13.9	4.98	270	311.35	[98, 1550]

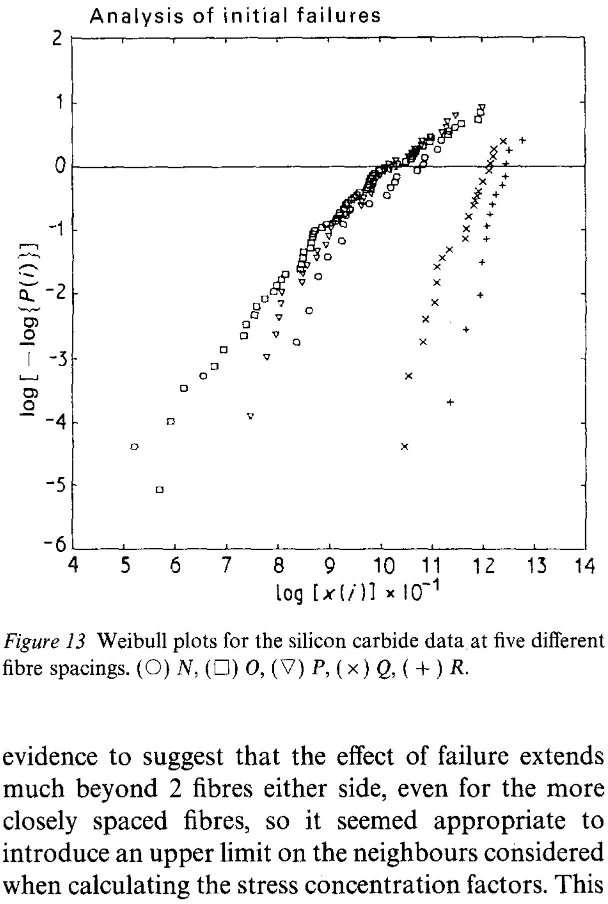


Figure 13 Weibull plots for the silicon carbide data at five different fibre spacings. (○) N, (□) O, (▽) P, (×) Q, (+) R.

evidence to suggest that the effect of failure extends much beyond 2 fibres either side, even for the more closely spaced fibres, so it seemed appropriate to introduce an upper limit on the neighbours considered when calculating the stress concentration factors. This is inevitably an approximation, but more realistic than no limit at all. The limit takes the form of a maximum number of neighbours to one side of a fibre and is denoted NLS in further results. The values used are those suggested by Clarke. Even though there is not expected to be load-sharing in the Q and R cases, a minimum value for NLS must be included in order to make inference about F. The results in Table VII show that this improves the likelihood.

TABLE VII

Data	NLS	w	α	$\hat{F}$	Min. − log$L$	Conf. int.
N	2	8.6	5.38	11	548.64	[9, 15]
O	2	6.1	6.85	25	1508.38	[18, 38]
P	1	7.9	5.57	101	1146.81	[58, 212]
Q	1	13.9	4.98	184	280.31	[52, 3000]
R	1	13.9	4.98	352	305.89	[89, 6000]

4. Extensions
4.1. Simultaneous failures revisited
It has usually been assumed that the probability of independent simultaneous failure is zero. Allowance may be made for this possibility, and will lead to some variation in the results. In particular, it becomes possible to produce confidence intervals which include infinity. The likelihood function, as constructed so far, does not permit this where simultaneous failures are involved. The assumption that there is an "initiating" failure automatically implies dependence of failures, i.e. F not equal to infinity, hence a finite F is always produced.

The assumption of one "initiating" failure raises another issue of particular relevance in the silicon carbide data. The interactions between breaks can be complicated, stretching over wide distances, dependent on whether regions of secondary failure overlap, positions of previously failed regions, and so on. It is perfectly possible that within a group of n failures recorded "simultaneously" there are independent sub-groups size n_i, where $\sum_i n_i = n$, $n_i \geqslant 1$. Up until now the data have been manually vetted and "fixed" so that the right sub-groups are taken, but this is hardly satisfactory.

The algorithm for calculating the probability of a group failure is as described in Wolstenholme [17], and a remedy for the above problem is simply to allow the initiating failure "group" to be of size k, $1 \leqslant k \leqslant n$. This also, as a by-product, allows F to tend to infinity, where appropriate, by effectively allowing all $n_i = 1$. This does, however, raise the question of how to get the right "measure" for the probabilities.

Take the simple example of two adjacent elements failing simultaneously, ignoring any secondary failure for the moment. The likelihood contribution for progressive failure is

prob(fibre 1 fails, then fibre 2) + prob(fibre 2 fails, then fibre 1)

$$= k_1 f(k_1 x)\mathrm{d}x[S(k_2 x) - S(k_2' x)] + k_2 f(k_2 x)\mathrm{d}x[S(k_1 x) - S(k_1' x)] \quad (18)$$

but dx is simply a constant of proportion and is, therefore, generally omitted. If these failures were independent $k_i' = k_i$ and a zero probability would be returned, and the likelihood contribution would have

TABLE VIII

Data	NLS	$\hat{F}$	Min. $-\log L$	Conf. int.
N	2	11	1086.57	[9, 15]
O	2	24	3073.55	[18, 37]
P	1	114	2476.75	[62, 275]
Q	1	201	586.05	[53, ∞]
R	1	420	679.52	[93, ∞]

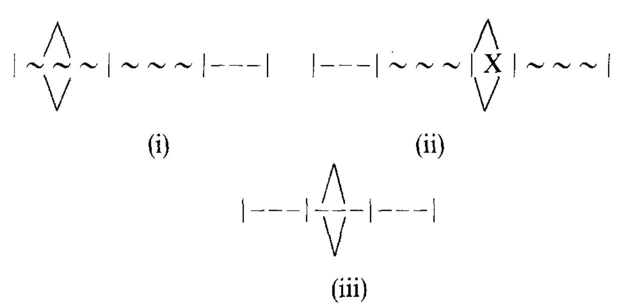
to be

$$k_1 f(k_1 x)\mathrm{d}x\, k_2 f(k_2 x)\mathrm{d}x \qquad (19)$$

If this possible independent failure is allowed automatically, the contribution must be the sum of Expressions 18 and 19, and must include dx otherwise the components contributing to the single likelihood term are not of the same measure. In effect this means contributing precisely $f(x)\mathrm{d}x$ to the likelihood wherever $f(x)$ arises. The question then is what value to use for dx, and in general it should reflect the accuracy of measurement of x. In the silicon carbide data the strain is calculated to the nearest 0.001%, so stress, x, is to the nearest 0.00001×347, and so dx is chosen to be 0.00347. The modified results are shown in Table VIII. As expected, the effect is small on the closer spacings where independent failures are unlikely, and introduces more uncertainty about F where load sharing is unlikely. (Note that the log-likelihood values are no longer comparable with earlier figures due to the scaling by dx.) The major advantage of this modification is that possible data dependences are now all automatically detected.

4.2. Treatment of resin cracks

A resin crack may occur where a fibre is already failed or damaged or in an as-yet unfailed region (see Fig. 6 for key)



In cases (i) and (ii) no likelihood contribution is appropriate for the element concerned because that will have been determined at the time of failure, but it is to be assumed that neighbouring elements will be under a changed stress concentration as a result. In case (iii), the fibre element concerned is probably unloaded, due to debonding of the matrix around the crack, so a survival probability should be assigned as the likelihood contribution based on the applied load and stress concentration factor as immediately prior to resin crack.

The effect of the resin crack on neighbouring fibres is unknown, but thought to be small. Arbitrarily, it was decided to make the contribution to r, the number of failed neighbours, increased by 0.5 if a resin crack is present. An increase in the possible status types for elements is required, so

$$\begin{aligned} t_{i,j} &= 0 = \text{failed} \\ &= 1 = \text{unfailed} \\ &= 2 = \text{failure and resin crack} \\ &= 3 = \text{resin crack only} \end{aligned} \qquad (20)$$

At any given point, it is still only those elements for which $t_{i,j} = 1$ which are "in play". For $t_{i,j} = 0$, the contribution to r is 1, for $t_{i,j} = 2$, the contribution is 1.5, and 0.5 if $t_{i,j} = 3$.

Analysis now follows of the complete N, O, and P data with the one exclusion of data set $O13$ which contained a large number of cracks and probably produces a large unknown effect. Table IX demonstrates the above modification. The results are to be compared with Table VIII, and perhaps are changed more than might be expected, in particular where the strain range is extended, though remain consistent in that the estimated F increases with fibre spacing.

The question does remain as to whether the single fibre strength distribution has been adequately described. As seen in Section 3.2.2 the analysis of initial failures produced some unexplained discrepancies in the 2, 4, and 6 diameter spaced arrays.

4.3. Bimodal failure and competing risks

Two types of failure are known to exist for this fibre, those initiated at the surface and those initiated at the tungsten core. Clarke examined the fracture surfaces of a sample of single fibres to establish the failure

TABLE IX

Data set		No. of arrays	No. of failures	Max. strain[a]	No. of cracks	$\hat{F}$	Conf. int.
N	pre-crack	4	106	0.908		11	[9, 15]
	total	4	153	1.012	14	22	[16, 34]
O	pre-crack	7	293	1.026		24	[18, 37]
	total	7	347	1.173	14	39	[27, 64]
P	pre-crack	4	247	1.126		114	[62, 275]
	total	5	328	1.126	1	87	[52, 177]

[a] This does not take into account any effective increase due to load-sharing.

mode, with the following results:

53 fibres in air, length 10 mm fibres:

surface 43 strain range 0.3%–0.92%

core 12 strain range 0.92%–1.12%

30 fibres, length 50 mm, in $\sim 1/2$ diameter resin coating:

surface 10 strain range 0.46%–0.96%

core 20 strain range 0.96%–1.1%

It would appear that the core-initiated failures occur in a very concentrated strain range, and that resin coating considerably inhibits surface-initiated failures. The 10-diameter spaced arrays (*R*) have failures occurring at 0.898%, 0.926%, 0.952%, 0.956%, 0.956%, etc. strain, and it seems highly likely that these are overwhelmingly core-initiated failures. It is possible that at the closer spacings the resin matrix has not inhibited the surface flaws to the extent predicted. Minimum strains of the order 0.5%–0.6% indicate that these are surface failures, but as strain increases, a bimodal failure distribution would seem appropriate. Clarke showed that a competing-risks model given by

$$S(x) = S^{(1)}(x)S^{(2)}(x) \tag{21}$$

where $S^{(1)}(x)$, $S^{(2)}(x)$ are, respectively, the Weibull survival functions for surface and core-initiated failures, fitted the above single fibre data better than a single mode Weibull. The assumption is made that these different modes of failure are independent.

A competing-risks distribution of this form may be used with both the chain-of-bundles and dependent bundles models, as weak-link scaling still applies, i.e.

$$\begin{aligned} S_L(x) &= S_L^{(1)}(x)S_L^{(2)}(x) \\ &= [S_1^{(1)}(x)]^L[S_1^{(2)}(x)]^L \end{aligned} \tag{22}$$

so

$$[S_L(x)]^{1/L} = S_1^{(1)}(x)S_1^{(2)}(x) \tag{23}$$

However, a joint estimation over the 5 parameters F, $w^{(1)}$, $\alpha^{(1)}$, $w^{(2)}$, $\alpha^{(2)}$, is neither practical or reliable as problems of identifiability arise, as discussed by Basu and Klein [20] for example. Such problems are rectifiable, where, for example, the failure type is known and then the marginal distribution for each failure type may be estimated separately, as in Kalbfleisch and Prentice [12].

Clarke quotes for the 50 mm resin-coated fibres the following Weibull parameters:

Total 30, $w = 6.4$, $\alpha = 6.73$, from a single mode Weibull plot
Surface 10, $w = 4.4$, $\alpha = 10.875$,
Core 20, $w = 31.9$, $\alpha = 4.205$.

The *O* data (4-diameter spacing) have the largest number of failures contributing to the initial failure analysis, and with $w = 6.1$ and $\alpha = 6.848$ would seem rather similar to the above. The values are also common to the *N*, *O* and *P* confidence regions in Fig. 12. On this basis, a competing risks strength distribution with the above parameters was tried, but produced negative log-likelihood values very significantly greater than those given by the single mode Weibull.

It has been suggested that the single embedded fibres and widely spaced fibres in arrays would display predominantly core-initiated failures, but the parameters $w = 13.9$ and $\alpha = 4.98$ give firstly, a greater spread of strength, and secondly, a lower characteristic strength at 200 mm than that given by $w = 31.9$ and $\alpha = 4.205$.

5. Conclusion

Whilst uncertainties may persist with regard to fibre strength, the dependent bundles model seems to perform well in estimating load-sharing parameters, provided a strength distribution which has the weakest-link property approximately applies. The estimated single mode Weibull based on the initial failures of each fibre in the arrays has in fact consistently given optimum likelihood values in the estimation of F, and without further precise experimental evidence would appear to be a satisfactory procedure.

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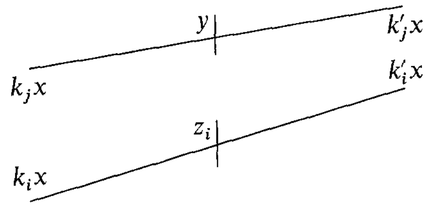
Appendix

An alternative definition for $g_i(y)$ in 2.6 is now outlined. It will be assumed that as y varies linearly between $k_{2,j}x$ and $k'_{2,j}x$ that the effective stress on elements about to be secondary failed varies linearly between $k_{2,i}x$ and $k'_{2,i}x$ – see Fig. 4.

For convenience, the fibre number subscript is omitted here.



Now

$$\frac{y - k_j x}{k'_j x - k_j x} = \frac{z_i - k_i x}{k'_i x - k_i x} \tag{A1}$$

therefore

$$\begin{aligned} z_i &= \frac{(k'_i - k_i)(y - k_j x)}{(k'_j - k_j)} + k_i x \\ &= \lambda_i (y - k_j x) + k_i x \end{aligned} \tag{A2}$$

where

$$\lambda_i = \frac{k'_i - k_i}{k'_j - k_j} \tag{A3}$$

Integral 1 (Section 2.6) becomes

$$\int_{k_j x}^{k'_j x} \prod_{i \neq j} S(\lambda_i(y - k_j x) + k_i x) f(y)\,\mathrm{d}y$$

$$= \int_{k_j x}^{k'_j x} \exp\{-\sum_i [\lambda_i(y - k_j x) + k_i x]^w \alpha^{-w}\}$$

$$\times w\alpha^{-w} y^{w-1}\,\mathrm{d}y \qquad \text{(A4)}$$

$$= \int_{k_j}^{k'_j} \exp\{-\sum_i [\lambda_i(ux - k_j x) + k_i x]^w \alpha^{-w}\}$$

$$\times w\alpha^{-w}(ux)^{w-1} x\,\mathrm{d}u \qquad y = ux$$

$$= w(x/\alpha)^w \int_{k_j}^{k'_j} \exp\{-x^w[\lambda_i(u - k_j) + k_i]^w\}$$

$$\times u^{w-1}\,\mathrm{d}u \qquad \text{(A5)}$$

Fig. A1 is a simple example where a group failure is completely surrounded by unfailed nearest neighbours.

Let x be the primary failure stress. In the case when fibre 1 is the initiating failure, the joint probability is $f(x)\,(\mathrm{d}x)\,[S(x)]^{m-1}\,[S(x)]^{j_1}$ multiplied by the contribution from fibre 2. The latter term is based on the primary failure in fibre 2 occurring somewhere between x and kx. Because the overlap region has no failed neighbouring elements, the fibre 2 primary failure stress y must be the survival stress for the consequent secondary failures, i.e. in the above analysis $z_i = y$, i.e. $\lambda_i = 1$ for the overlap elements, and for the j_2 elements not in the overlap, which are unaffected by the failure in 1, $z_i = k_i x$, i.e. $\lambda_i = 0$. So Equation A4 becomes

$$[S(x)]^{j_2} \int_x^{kx} [S(y)]^{m-1} f(y)\mathrm{d}y \qquad \text{(A6)}$$

Assuming a Weibull distribution for element strength with parameters w and α, the integral in Expression A6 becomes

$$\int_x^{kx} w/y(y/\alpha)^w \exp[-m(y/\alpha)^w]\mathrm{d}y$$

$$= \{\exp[-m(x/\alpha)^w] - \exp[-m(kx/\alpha)^w]\}/m \qquad \text{(A7)}$$

i.e. this is the probability that a fibre length m fails somewhere between stress x and kx, scaled by a factor

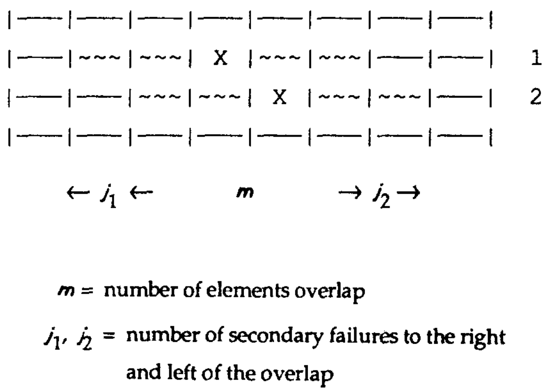


Figure A1 Example of a group failure surrounded by unfailed elements in the nearest neighbours.

TABLE X

Data	NLS	$\hat{F}$	Min. $-\log L$	Conf. int.
N	2	11	1086.82	[9, 15]
O	2	25	3073.87	[18, 38]
P	1	114	2476.83	[62, 279]
Q	1	201	586.05	[53, ∞]
R	1	420	679.52	[93, ∞]

m, which is due to the failing element being fixed at one of the m units. Such scaling factors are in fact relevant to any m element contribution, but nothing is lost with regard to inference about parameters of interest if the scalars are omitted.

The integral in Expression A5 has to be evaluated numerically, and for that purpose here Gauss–Legendre polynomials of order 4 were used [21]. Because all the survival probability components gradually decrease over the integration, the probabilities are smaller and hence lead to a smaller log-likelihood than before, but the likelihood tends to the same limits as before as F tends to zero or infinity. This, in general, leads to wider confidence intervals for F.

For the silicon carbide data, the incidence of simultaneous failure is relatively small, and the survival probability components are always close to one, so the net effect is very small, as demonstrated in Table X. The results are to be compared with Table VIII.

A limited analytic comparison can be made between these approaches, and it can be shown that because the survivor function terms contributing to the likelihood are frequently of the order 0.95–0.98, the two approaches produce a likelihood function of similar behaviour for values of F greater than 10, approximately.

Other monotonically increasing paths from kx to $k'x$ may be described in Fig. 4, but investigations for these data have again shown the effects on the results to be small. It is to be expected that the technology of experimentation will increasingly reduce the incidence of failures recorded simultaneously, thus making the treatment of this event less of an issue.

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